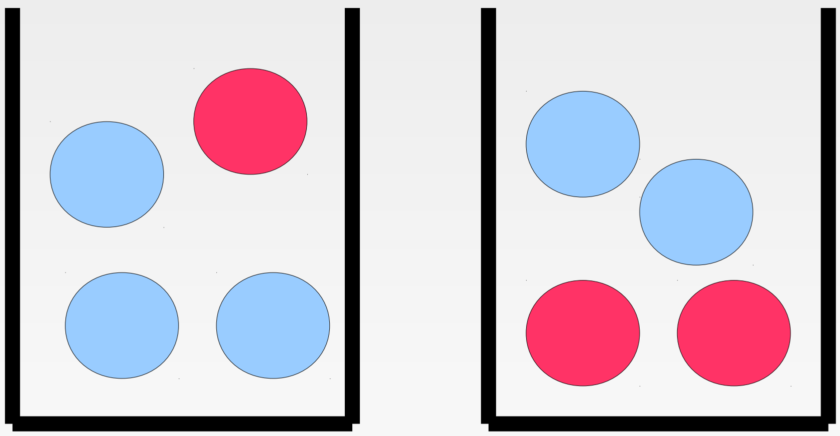
Methods in Combinatorics

- Finite, Countable, Discrete.
  - Enumerating objects which satisfy a condition.
- In short: Counting Stuff

  → Balls & Urns



12-fold Way

Balls	Urns	unrestricted	≤ 1	≥ 1
labeled	labeled	u^b	$(u)_b$	$u!S(b,u)$
unlabeled	labeled	$\left(\binom{u}{b}\right)$	$\binom{u}{b}$	$\left(\binom{u}{b-u}\right)$
labeled	unlabeled	$\sum_{i=1}^{u} S(b,i)$	$[b \leq u]$	$S(b,u)$
unlabeled	unlabeled	$\sum_{i=1}^{u} p_i(b)$	$[b \leq u]$	$p_u(b)$

Let *u* represent the number of available urns and *b* the number of balls.

≤ 1 : No more than one ball per urn

≥ 1 : At least one ball per urn

12-fold Way

- Labeled balls, labeled urns, unrestricted:
  - Total of u^b possibilities.
- Labeled balls, labeled urns, ≤ 1:
  - Total of $(u)_b = \mathrm{P}(u,b) = \dfrac{u!}{(u-b)!}$ possibilities.
- Unlabeled balls, labeled urns, ≤ 1:
  - Total of $\binom{u}{b} = \mathrm{C}(u,b) = \dfrac{u!}{b!(u-b)!}$ possibilities.
  - Pascal's Triangle
  - Binomial Coefficients

12-fold Way

- Unlabeled balls, labeled urns, unrestricted:
  - Notation: $$\left(\!\!\binom{u}{b}\!\!\right) = \binom{u+b-1}{b} = \frac{(u+b-1)!}{b!(u-1)!}$$
  - Stars-and-Bars Method:

    Imagine the *u* urns as spaces between the *u-1* bars.

    Example: (4 balls assigned to 3 urns)

    ★ | ★ ★ ★ |
  - Among *b+u-1* symbols we choose *b* to be stars...

12-fold Way

- Unlabeled balls, labeled urns, ≥ 1
  - Put one ball in each urn. Now there are $b - u$ balls that can be distributed without restriction and so this is the previous case.
  - Total:

$$\left(\!\!\binom{u}{b-u}\!\!\right) = \binom{u-1}{b-1}$$

12-fold Way [Stirling Numbers]

- Labeled balls, unlabeled urns, ≥ 1
  - Stirling numbers of the second kind. $S(b,u) = S_b^{(u)} = \left\{ \begin{matrix} b \\ u \end{matrix} \right\}$
  - → S(n, k) is defined to be the number of ways to partition n objects into k non-empty, unordered sets.

Example: S(4, 2) = 7

(since {1,2,3,4} can be partitioned into 2 sets in 7 ways as follows:)

{1} ∪ {2, 3, 4}, {2} ∪ {1, 3, 4}, {3} ∪ {1, 2, 4}, {4} ∪ {1, 2, 3},

{1, 2} ∪ {3, 4}, {1, 3} ∪ {2, 4}, {1, 4} ∪ {2, 3}

  - Bell numbers are the total partitions for b (i.e. u goes from 0 to b)

- How to calculate? No explicit formula. Recurrence relation:

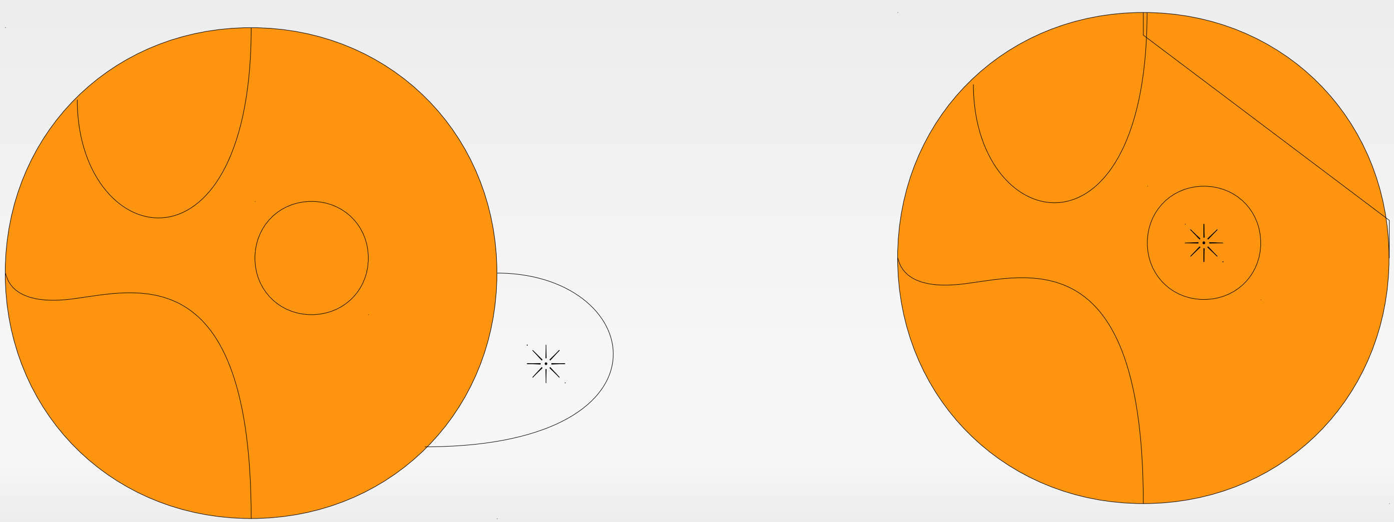
$$S(n, k) = k\, S(n-1, k) + S(n-1, k-1)$$

12-fold Way [Stirling Numbers]

- Labeled balls, unlabeled urns, ≥ 1
  - Stirling numbers of the second kind. $S(n,k) = S_n^{(k)} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$

- How to calculate? No explicit formula. Recurrence relation:

$$S(n, k) = k\, S(n - 1, k) + S(n - 1, k - 1) \text{ where } S(n,n) = S(n,1) = 1$$



Catalan Numbers

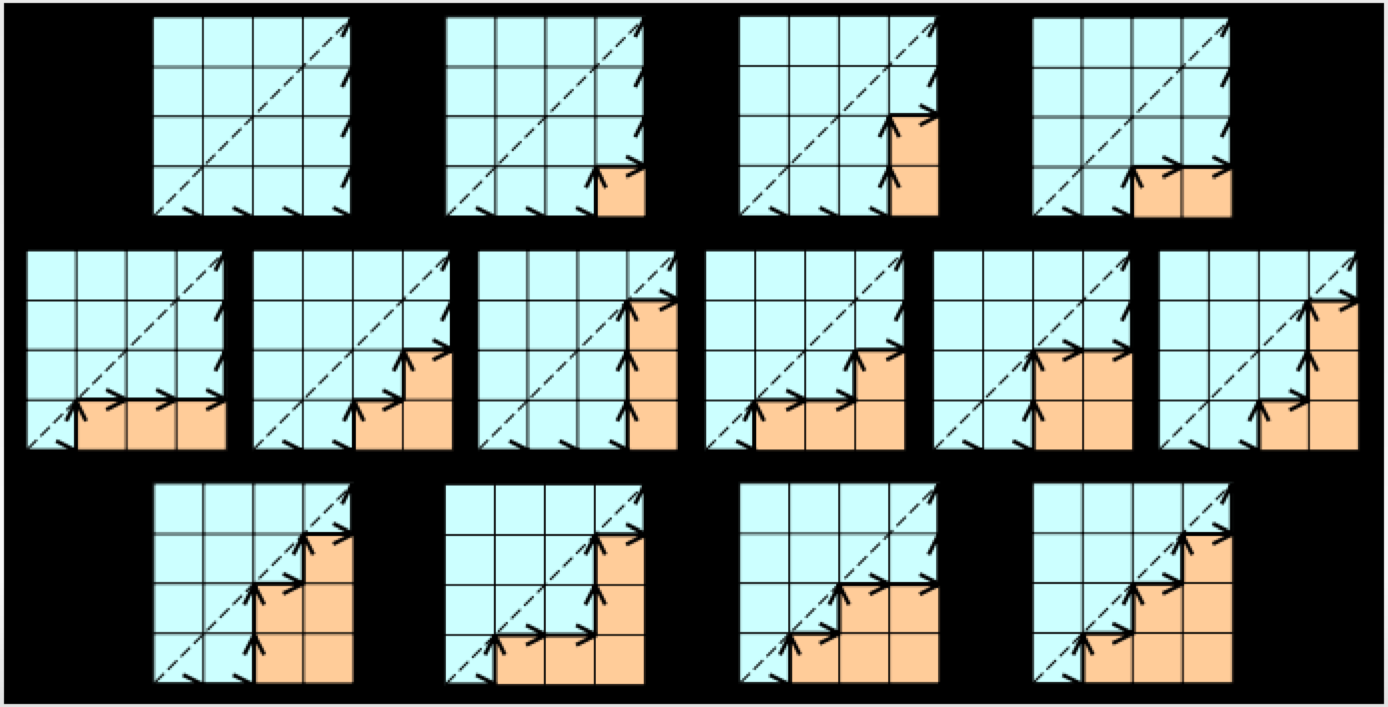
- Many applications.

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

  - # sequences with correctly matched parenthesis for n pairs:

    ((())) ()(()) ()()() (())() (()())

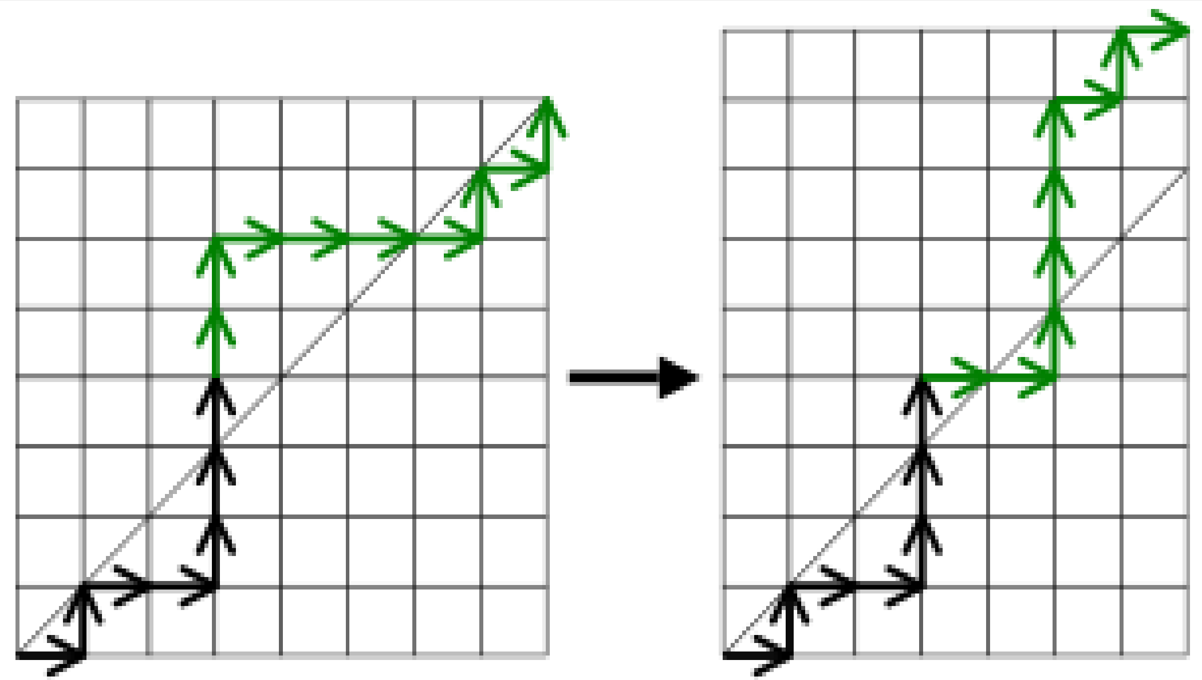
  - # monotonic paths not crossing the diagonal of an nxn grid:



Catalan Numbers

- Many applications.

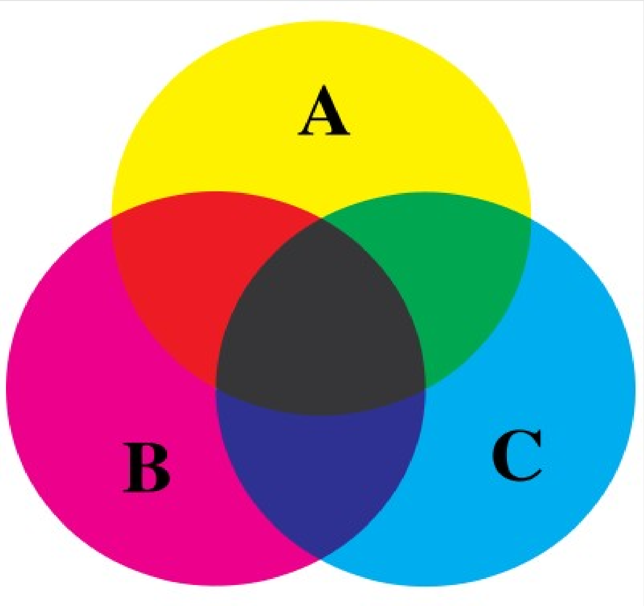
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$$



Inclusion Exclusion Principle

- ~ Common Sense

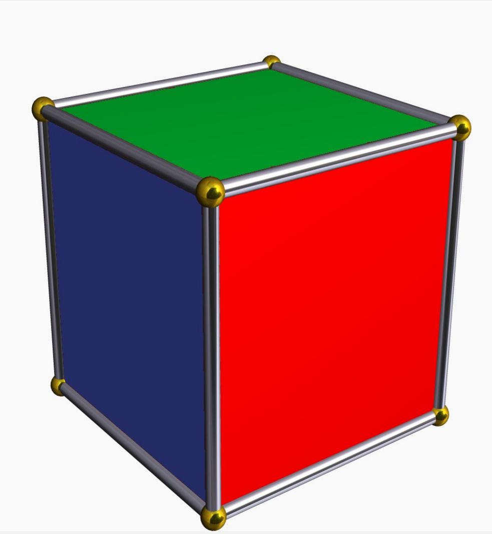
Compensate for over counting when evaluating the cardinality of the union of finite sets.



$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j\,:\,1 \le i < j \le n} |A_i \cap A_j| + \sum_{i,j,k\,:\,1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .$$

Burnside's Lemma

- Colourings which are invarient under transformation.
  - E.g. colouring the faces of a cube with 3 colours:



  - one identity element
  - six 90-degree face rotations
  - three 180-degree face rotations
  - eight 120-degree vertex rotations
  - six 180-degree edge rotations

    - How many faces remain unchanged after each transformation?

Total # of possibilites: $\frac{1}{24}\left(3^{6}+6\times 3^{3}+3\times 3^{4}+8\times 3^{2}+6\times 3^{3}\right)$